Generalizations of some integrals over the unitary group

B Schlittgen[1] **and T Wettig**[1,2]

[1] Department of Physics, Yale University, New Haven, CT 06520-8120, USA
[2] RIKEN-BNL Research Center, Upton, NY 11973-5000, USA

Abstract
Using the character expansion method, we generalize several well-known
integrals over the unitary group to the case where general complex matrices
appear in the integrand. These integrals are of interest in the theory of random
matrices and may also find applications in lattice gauge theory.

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1. Introduction

Physical systems described by non-Hermitian operators have recently attracted a lot of attention
in the literature. Applications range from flux-line pinning in superconductors [1] over
dissipation and scattering in quantum chaos [2] to quantum chromodynamics (QCD) at non-
zero density [3]. In turn, the interest in these applications has stimulated new mathematical
studies of non-Hermitian matrices, in particular, in the field of random matrix theory.

We have been led to consider non-Hermitian matrices in our work on the colour–flavour
transformation [4–6]. This transformation involves an integration over complex matrices Z
without any symmetry requirements. Applying the transformation to lattice QCD [6–8] results
in a complex action that is not amenable to standard Monte Carlo algorithms. A possible way
of trying to resolve this problem is to write $Z = HU$ with H Hermitian and U unitary, and
to perform the integral over U analytically so that only the integration over H has to be done
numerically. This approach leads to the following integral over the unitary group,

$$\mathcal{I}_1 = \int_{U(N)} \mathrm{d}\mu(U) \det^\nu U \, \mathrm{e}^{\frac{1}{2}\,\mathrm{tr}(AU+BU^\dagger)} \tag{1}$$

where $\mathrm{d}\mu(U)$ is the Haar measure of $U(N)$, ν is an integer, which without loss of generality
we take to be non-negative, and $A, B \in \mathrm{Gl}(N, \mathbb{C})$. The above integral is the main focus of this
paper. It is well known for $A = B^\dagger$ [9–11] (where it leads to the effective partition function of
QCD in the Leutwyler–Smilga regime [10, 12]), but to the best of our knowledge $\mathcal{I}_1$ had not
been computed before for $A \neq B^\dagger$. We found that the latter case can be solved rather easily
using the character expansion method put forward by Balantekin [13, 14]. Although the case

of non-integer ν appears to be beyond this approach, we expect our result to hold in that case as well. Employing the same method, we could also compute the integral which arises from the factorization $Z = U \Lambda V^\dagger$ in the colour–flavour transformed action mentioned above. The matrices U and V are unitary, while Λ is diagonal with non-negative entries. The resulting integral in slightly generalized form is given by

$$\mathcal{I}_2 = \int_{U(N)} \mathrm{d}\mu(U) \int_{U(N)} \mathrm{d}\mu(V) \det^\nu(UV) \, \mathrm{e}^{\frac{1}{2} \operatorname{tr}(UAVB + CV^\dagger DU^\dagger)} \tag{2}$$

where $A, B, C, D \in \mathrm{Gl}(N, \mathbb{C})$. This integral was previously known only for the case of $\nu = 0$, $C = B^\dagger$ and $D = A^\dagger$ [10, 15, 16]. We also show that the generalization of $\mathcal{I}_2$ to the case of unequal dimensions of U and V leads to an integral which can be non-zero only if determinants of U and V are absent from the integrand. We conjecture an expression for the result in this case.

In section 2, we derive results for $\mathcal{I}_1$ and $\mathcal{I}_2$. In addition, we discuss $\mathcal{I}_2$ with unequal dimensions of U and V and briefly consider the (trivial) generalization of the Itzykson–Zuber integral to the case of general complex matrices in the integrand. We also comment on the applicability of our results to the case of non-invertible matrices. Conclusions are drawn in section 3.

2. Calculation of the group integrals

Our calculations rely on the formalism of [14] and on several examples worked out in detail therein. Rather than reproducing the material from that work, we will refer to the relevant equations in [14] whenever appropriate. Thus, the reader is advised to have a copy of this reference at hand.

The key observation for the generalization of the results of [14] is that the representation theories of the groups $U(N)$ and $\mathrm{Gl}(N)$ are essentially the same. In particular, Weyl's character formula

$$\chi_r(X) = \frac{\det\left(x_i^{n_j + N - j}\right)}{\Delta(x_1, \ldots, x_N)} \tag{3}$$

holds not just for unitary, but also for general linear matrices, cf [17]. Here, $r = (n_1, \ldots, n_N)$ denotes an irreducible representation of $\mathrm{Gl}(N)$ labelled by non-negative, non-increasing integers. The x_i ($i = 1, \ldots, N$) are the eigenvalues of the matrix X.

2.1. Calculation of $\mathcal{I}_1$

Using equation (3.5) of [14], we write

$$\det^\nu(AU) \, \mathrm{e}^{\operatorname{tr} AU} = \sum_r \alpha_r^{(\nu)} \chi_r(AU) \tag{4}$$

$$\mathrm{e}^{\operatorname{tr} BU^\dagger} = \sum_r \alpha_r^{(0)} \chi_r(BU^\dagger). \tag{5}$$

Here, the sums are over all irreducible representations of $\mathrm{Gl}(N)$ labelled by $r = (n_1, n_2, \ldots, n_N)$ as above. The corresponding characters are denoted by χ_r. For convenience, we have left out the factor of $\frac{1}{2}$ in the exponent of equation (1), which will be reinstated in the final result. The coefficients in the character expansion are given by

$$\alpha_r^{(\nu)} = \det\left[\frac{1}{(n_j - \nu + i - j)!}\right] \tag{6}$$

where i and j run from 1 to N, labelling the rows and columns of the matrix. We thus obtain

$$\hat{\mathcal{I}}_1 \equiv \int_{U(N)} d\mu(U) \det^{\nu}(AU) e^{\text{tr}(AU+BU^{\dagger})}$$

$$= \sum_r \sum_{r'} \alpha_r^{(\nu)} \alpha_{r'}^{(0)} \int_{U(N)} d\mu(U) \chi_r(AU) \chi_{r'}(BU^{\dagger}). \tag{7}$$

Now, we can write

$$\chi_r(AU) \chi_{r'}(BU^{\dagger}) = A_{ab}^{(r)} U_{ba}^{(r)} B_{cd}^{(r')} U_{cd}^{(r')*} \tag{8}$$

where the superscript serves as a reminder that these matrices live in the representations r and r' of $\text{Gl}(N)$, respectively. If we restrict ourselves to unitary matrices, we have corresponding irreducible representations of the subgroup $U(N)$ of $\text{Gl}(N)$, and hence, we can use the group theoretical result

$$\int_{U(N)} d\mu(U) U_{ba}^{(r)} U_{cd}^{(r')*} = \frac{1}{d_r} \delta^{rr'} \delta_{bc} \delta_{ad} \tag{9}$$

where d_r is the dimension of r, regardless of whether we consider it as a representation of $\text{Gl}(N)$ or $U(N)$. It then follows that

$$\hat{\mathcal{I}}_1 = \sum_r \frac{\alpha_r^{(0)}}{d_r} \alpha_r^{(\nu)} \chi_r(AB). \tag{10}$$

From equations (3.5), (3.3) and (2.9) of [14], we obtain

$$\frac{\alpha_r^{(\nu)}}{d_r} = \prod_{i=1}^N \frac{(N-i)!}{(k_i-\nu)!} \qquad \text{with} \quad k_i = N + n_i - i. \tag{11}$$

The matrix AB has N eigenvalues which we denote by $\mu_1^2, \ldots, \mu_N^2$. We now apply Weyl's formula,

$$\chi_r(AB) = \frac{\det\left[\mu_i^{2(n_j+N-j)}\right]}{\Delta(\mu^2)} \tag{12}$$

where

$$\Delta(x) = \prod_{i<j}^N (x_i - x_j) \tag{13}$$

is the Vandermonde determinant. Inserting this expression, together with equations (6) and (11), into equation (10) yields

$$\hat{\mathcal{I}}_1 = \left[\prod_{n=1}^{N-1} n!\right] \frac{1}{\Delta(\mu^2)} \sum_r \det\left[\frac{1}{k_j!(k_j-N-\nu+i)!}\right] \det\left[\mu_i^{2k_j}\right]. \tag{14}$$

Applying the Binet–Cauchy formula, see equation (B4) of [14], and using the power series expansion of the Bessel function,

$$\frac{I_{\lambda}(2y)}{y^{\lambda}} = \sum_{k=0}^{\infty} \frac{y^{2k}}{k!(k+\lambda)!} \tag{15}$$

we obtain immediately

$$\hat{\mathcal{I}}_1 = \left[\prod_{n=1}^{N-1} n!\right] \frac{1}{\Delta(\mu^2)} \det\left[\mu_j^{N-i+\nu} I_{i-N-\nu}(2\mu_j)\right]. \tag{16}$$

Pulling out the factors of μ_j^ν and rearranging the determinant using $I_n = I_{-n}$ yields

$$\det\left[\mu_j^{N-i+\nu} I_{i-N-\nu}(2\mu_j)\right] = \det^{\frac{\nu}{2}}(AB)\,\det\left[\mu_j^{N-i} I_{i-N-\nu}(2\mu_j)\right]$$

$$= \det^{\frac{\nu}{2}}(AB)\,\det\left[\mu_j^{i-1} I_{\nu+i-1}(2\mu_j)\right]. \tag{17}$$

Finally, we rescale A and B by $\frac{1}{2}$ to reinstate the factor of $\frac{1}{2}$ in the exponent of equation (1) to obtain

$$\int_{U(N)} d\mu(U)\,\det^\nu U\,e^{\frac{1}{2}\,\text{tr}(AU+BU^\dagger)} = 2^{\frac{N(N-1)}{2}}\left[\prod_{n=1}^{N-1} n!\right]\left(\frac{\det B}{\det A}\right)^{\frac{\nu}{2}} \frac{\det\left[\mu_i^{j-1} I_{\nu+j-1}(\mu_i)\right]}{\Delta(\mu^2)} \tag{18}$$

where, once again, the μ_i^2 are the eigenvalues of AB.

2.2. Calculation of $\mathcal{I}_2$

Note first that in the integrand of $\mathcal{I}_2$, the determinants of U and V have to be raised to the same power, as indicated in equation (2); otherwise, the integrations over the $U(1)$ subgroups of U and V simply render $\mathcal{I}_2$ zero.

Using again equation (3.5) of [14], we have

$$\det^\nu(UAVB)\,e^{\text{tr}\,UAVB} = \sum_r \alpha_r^{(\nu)} \chi_r(AUVB) \tag{19}$$

$$e^{\text{tr}\,CV^\dagger DU^\dagger} = \sum_r \alpha_r^{(0)} \chi_r(CV^\dagger DU^\dagger) \tag{20}$$

with $\alpha_r^{(\nu)}$ given in equation (6). Thus,

$$\hat{\mathcal{I}}_2 \equiv \int_{U(N)} d\mu(U) \int_{U(N)} d\mu(V)\,\det^\nu(UAVB)\,e^{\text{tr}(UAVB+CV^\dagger DU^\dagger)}$$

$$= \sum_r \sum_{r'} \alpha_r^{(\nu)} \alpha_{r'}^{(0)} \int_{U(N)} d\mu(U) \int_{U(N)} d\mu(V) \chi_r(UAVB) \chi_{r'}(CV^\dagger DU^\dagger)$$

$$= \sum_r \frac{\alpha_r^{(\nu)} \alpha_r^{(0)}}{d_r} \int_{U(N)} d\mu(U) \chi_r(BUADU^\dagger C)$$

$$= \sum_r \frac{\alpha_r^{(\nu)} \alpha_r^{(0)}}{d_r^2} \chi_r(AD) \chi_r(BC) \tag{21}$$

where we have made use of equations (8) and (9). From equation (11) we have

$$\frac{\alpha_r^{(\nu)} \alpha_r^{(0)}}{d_r^2} = \left[\prod_{n=1}^{N-1} n!\right]^2 \prod_{i=1}^{N} \frac{1}{k_i!(k_i-\nu)!}. \tag{22}$$

Now denote the eigenvalues of the matrices AD and BC by $x_1^2, \ldots, x_N^2$ and $y_1^2, \ldots, y_N^2$, respectively. Using again Weyl's formula (12), equation (21) thus becomes

$$\hat{\mathcal{I}}_2 = \left[\prod_{n=1}^{N-1} n!\right]^2 \frac{1}{\Delta(x^2)\Delta(y^2)} \sum_r \prod_{i=1}^{N} \frac{1}{k_i!(k_i-\nu)!}\,\det\left[x_i^{2k_j}\right]\,\det\left[y_i^{2k_j}\right]. \tag{23}$$

Employing the expansion theorem given in equation (B2) of [14] and noting equation (15) again, this yields

$$\hat{\mathcal{I}}_2 = \left[\prod_{n=1}^{N-1} n!\right]^2 \frac{1}{\Delta(x^2)\Delta(y^2)}\,\det[f(x_i y_j)] \tag{24}$$

with

$$f(z) = z^\nu I_{-\nu}(2z).$$ (25)

Pulling the factors of x_i^ν and y_j^ν out of the determinant and using $I_n = I_{-n}$, we obtain

$$\hat{\mathcal{I}}_2 = \left[\prod_{n=1}^{N-1} n! \right]^2 \det^{\frac{\nu}{2}}(ABCD) \frac{\det[I_\nu(2x_i y_j)]}{\Delta(x^2)\Delta(y^2)}.$$ (26)

We finally rescale A, B, C and D by $1/\sqrt{2}$ to reinstate the factor of $\frac{1}{2}$ in the exponent of equation (2) to obtain

$$\int_{U(N)} d\mu(U) \int_{U(N)} d\mu(V) \det^\nu(UV) e^{\frac{1}{2}\operatorname{tr}(UAVB+CV^\dagger DU^\dagger)}$$
$$= 2^{N(N-1)} \left[\prod_{n=1}^{N-1} n! \right]^2 \left(\frac{\det(CD)}{\det(AB)} \right)^{\frac{\nu}{2}} \frac{\det[I_\nu(x_i y_j)]}{\Delta(x^2)\Delta(y^2)}.$$ (27)

Once again, the x_i^2 and y_j^2 are the eigenvalues of AD and BC, respectively.

Let us now consider the case in which U and V have different dimensions, i.e.

$$\mathcal{I}_2^{(N,M)} = \int_{U(N)} d\mu(U) \int_{U(M)} d\mu(V) e^{\frac{1}{2}\operatorname{tr}(UAVB+CV^\dagger DU^\dagger)}.$$ (28)

In this case, A and C are complex $N \times M$ matrices, and B and D are complex $M \times N$ matrices. For definiteness, we shall take $M < N$.

At this point, we have not been able to prove a result for $\mathcal{I}_2^{(N,M)}$, but we conjecture, based on explicit calculations for small N and M, as well as on numerical experimentation, that the result takes the form

$$\mathcal{I}_2^{(N,M)} = 2^{M(N-1)} \left[\prod_{n=N-M}^{N-1} n! \right] \left[\prod_{m=N-M}^{M-1} m! \right] \frac{\det[I_{N-M}(x_i y_j)]}{\Delta(x^2)\Delta(y^2) \prod_{i=1}^{M}(x_i y_i)^{N-M}}.$$ (29)

Here, x_i^2 and y_i^2 ($i = 1, \ldots, M$) denote the (non-zero) eigenvalues of DA and BC, respectively. This expression also reduces to the well-known result in the case where $C = B^\dagger$ and $D = A^\dagger$, cf [15, 10, 16].

Note that we have not included any determinant terms in the integrand of $\mathcal{I}_2^{(N,M)}$. If we included, say, $\det^\nu U \det^\eta V$ in the integrand, integrations over the $U(1)$ subgroups of $U(N)$ and $U(M)$ show immediately that the value of the resulting integral is zero unless, possibly, ν and η are related by $N\nu = M\eta$. We now show that even if this relation holds, the integral gives zero for any $\nu \neq 0$, and hence also for $\eta \neq 0$. To see this, suppose that $\nu \neq 0$, and let us perform the integral of U over $U(N)$, leaving the integral over $U(M)$ untouched for the moment. The result could be read off from equation (18) if the matrices AVB and $CV^\dagger D$ were $\mathrm{Gl}(N)$ matrices. However, since $M < N$, these matrices are not of full rank, and therefore $N - M$ of their eigenvalues are equal to zero. Except on a set of measure zero they have the same rank, so that a limiting process leads to a finite value of $\det(CV^\dagger D)/\det(AVB)$, given by $\det(V^\dagger DC)/\det(VBA)$. We then need to find the limit of $\det\left[\mu_i^{j-1} I_{\nu+j-1}(\mu_i)\right]/\Delta(\mu^2)$ as $\mu_{M+1}, \ldots, \mu_N \to 0$. In this context, $\mu_1^2, \ldots, \mu_M^2$ denote the non-zero eigenvalues of $AVBCV^\dagger D$. In fact, letting just $\mu_N \to 0$, it is easy to see that the above expression goes to zero, unless $\nu = 0$, which shows that $\mathcal{I}_2^{(N,M)} = 0$, unless $\nu = \eta = 0$.

2.3. Generalization of the Itzykson–Zuber integral

The integral

$$\mathcal{I}_3 = \int_{U(N)} \mathrm{d}\mu(U)\, \mathrm{e}^{\mathrm{tr}(AUBU^\dagger)} = \left[\prod_{n=1}^{N-1} n! \right] \frac{\det[\exp(x_i y_j)]}{\Delta(x)\Delta(y)} \tag{30}$$

was computed in [18] for the case where A and B are Hermitian matrices with real eigenvalues $x_1, \ldots, x_N$ and $y_1, \ldots, y_N$, respectively. This is a special case of a more general result due to Harish-Chandra [19].

Following the calculation of this integral in [14], it is immediately obvious that the only change in the final result is the replacement of the eigenvalues of the Hermitian matrices A and B by the eigenvalues of their general complex versions.

Also, including the determinant of U in the integrand (raised to any non-zero power) would give zero due to the integration over the $U(1)$ subgroup.

2.4. Comment on non-invertible matrices

In deriving the above results, we have assumed that the matrices A, B, C and D are elements of $\mathrm{Gl}(N, \mathbb{C})$. However, the integrals $\mathcal{I}_1$ through $\mathcal{I}_3$ exist even if the matrices on which they depend are not of full rank. In this case, we can consider the limit in which one or more of the eigenvalues of the matrix approach zero. Uniform convergence permits the interchange of this limit and the integration over the unitary group. A l'Hôpital procedure on the right-hand side of equation (18), (27) or (30) then leads to a finite (though possibly zero) result.

3. Conclusions

We have derived generalizations of several well-known integrals over the unitary group to the case where general complex matrices appear in the integrand. These integrals may find applications in lattice gauge theory but are also of purely mathematical interest, in particular in the theory of random matrices. As mentioned in the introduction, our motivation for studying these integrals originated from the complex action problem that arises if the colour–flavour transformation is applied to lattice QCD. We found that the integral $\mathcal{I}_1$ solved this complex action problem for one quark flavour but unfortunately not for two or more flavours.

Our results were obtained by a straightforward application of Balantekin's character expansion method. It would be interesting to investigate the feasibility of other well-known methods to compute integrals over the unitary group, such as the diffusion equation method, in the cases we have considered. It should also be possible to generalize the results of the present paper to integrals over the super-unitary group.

Acknowledgments

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